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Tridiagonal matrices, orthogonal polynomials and Diophantine relations: II

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Abstract
In this paper we apply the machinery of the previous paper of this series to results reported in our first paper devoted to these topics: in particular, we extend and prove certain Diophantine findings due to Christophe Smet reported there.

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1. Introduction and review of previous findings

Recently certain *Diophantine* conjectures, arrived at [1, 2] via the investigation of certain *isochronous* dynamical systems, were proven [3]. We were led by these developments to introduce a theoretical framework yielding several other *Diophantine* findings involving tridiagonal matrices and orthogonal polynomials [4]. Purpose and scope of the present paper is to show how this machinery is applicable to the polynomials introduced in our first paper devoted to these topics [3], and in particular to extend and prove some findings due to Christophe Smet reported there. Alternative proofs of the Smet findings [3]—including the demonstration of certain identities involving sums of binomial coefficients that we did not find in the standard compilations [5–7]—are provided in the appendix.

As indicated by the Roman numeral appended to its title, this paper is the second (or rather the third: the first paper was not numbered) of a series [3, 4] devoted to *Diophantine* relations, in particular to the identification of polynomials defined by three-term recursion relations—hence belonging to families of orthogonal polynomials—featuring *zeros* given by neat formulas involving *integers*, or equivalently of tridiagonal matrices which are remarkable in as much as their *eigenvalues* are likewise given by neat formulas involving *integers*.

We made an effort to render this paper self-contained, while also trying to avoid excessive repetitions of previous findings. Hence occasionally we employ below notations used in the papers [3, 4] without redefining them when they are self-evident, and we refer to equations of these two papers without explaining in detail their origin; yet we always report these equations in full, since by editorial policy we were required to revise this paper so that its reader may go through it without having to retrieve equations from our two previous papers [3, 4]. For this same reason we now review, in the last part of this introductory section, the main results of [4].

The orthogonal polynomials $p_n^{(\nu)}(x)$ on which we focus are generally identified by the three-term recursion relation [4, equation (1)],

$$p_{n+1}^{(\nu)}(x) = \left[x + a_n^{(\nu)}\right] p_n^{(\nu)}(x) + b_n^{(\nu)} p_{n-1}^{(\nu)}(x), \tag{1a}$$

$$p_0^{(\nu)}(x) = 1, \qquad p_1^{(\nu)}(x) = x + a_0^{(\nu)}. \tag{1b}$$

Here and hereafter the index n denotes the order of these polynomials in the variable x, while ν is another parameter associated with these polynomials.

Let us now report as propositions 1.1, 1.2 and 1.3 the three propositions 2.3, 2.4 and 2.8 of [4], and some of the corresponding *corollaries* and *remarks*.

Proposition 1.1. *Assume that the quantities $A_n^{(\nu)}$ and $\alpha^{(\nu)}$ satisfy the nonlinear recursion relation*

$$\left[A_{n-1}^{(\nu)} - A_{n-1}^{(\nu-1)}\right]\left[A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \alpha^{(\nu)}\right] = \left[A_{n-1}^{(\nu-1)} - A_{n-1}^{(\nu-2)}\right]\left[A_{n-1}^{(\nu-1)} - A_{n-2}^{(\nu-2)} + \alpha^{(\nu-1)}\right] \tag{2a}$$

with the boundary condition

$$A_0^{(\nu)} = A, \tag{2b}$$

where A is an arbitrary constant (independent of ν), and that the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are defined in terms of these quantities by the following formulae:

$$a_n^{(\nu)} = A_{n+1}^{(\nu)} - A_n^{(\nu)}, \tag{3a}$$

$$b_n^{(\nu)} = \left[A_n^{(\nu)} - A_n^{(\nu-1)}\right]\left[A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \alpha^{(\nu)}\right]. \tag{3b}$$

Then the polynomials $p_n^{(\nu)}(x)$ identified by the recursion relation (1) satisfy the following additional recursion relation (involving a shift both in the order n of the polynomials and in the parameter ν):

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)} p_{n-1}^{(\nu-1)}(x), \tag{4a}$$

with

$$g_n^{(\nu)} = A_n^{(\nu)} - A_n^{(\nu-1)}. \tag{4b}$$

Proposition 1.2. *If the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ are defined by the recursion relation (1) and the coefficients $b_n^{(\nu)}$ satisfy the relation*

$$b_n^{(n)} = 0, \tag{5}$$

entailing that, for $\nu = n$, the recursion relation (1) reads

$$p_{n+1}^{(n)}(x) = \left(x + a_n^{(n)}\right) p_n^{(n)}(x), \tag{6}$$

then there holds the factorization

$$p_n^{(m)}(x) = \tilde{p}_{n-m}^{(-m)}(x)\, p_m^{(m)}(x), \qquad m = 0, 1, \ldots, n, \tag{7}$$

with the 'complementary' polynomials $\tilde{p}_n^{(-m)}(x)$ *(of course of degree* n*) defined by the following three-term recursion relation analogous (but not identical) to (1):*

$$\tilde{p}_{n+1}^{(-m)}(x) = \left(x + a_{n+m}^{(m)}\right)\tilde{p}_n^{(-m)}(x) + b_{n+m}^{(m)}\,\tilde{p}_{n-1}^{(-m)}(x), \tag{8a}$$

$$\tilde{p}_{-1}^{(-m)}(x) = 0, \qquad \tilde{p}_0^{(-m)}(x) = 1, \tag{8b}$$

entailing

$$\tilde{p}_1^{(-m)}(x) = x + a_m^{(m)}, \tag{8c}$$

$$\tilde{p}_2^{(-m)}(x) = \left(x + a_{m+1}^{(m)}\right)\left(x + a_m^{(m)}\right) + b_{m+1}^{(m)} = \left(x - x_m^{(+)}\right)\left(x - x_m^{(-)}\right) \tag{8d}$$

with

$$x_m^{(\pm)} = \tfrac{1}{2}\left\{-a_m^{(m)} - a_{m+1}^{(m)} \pm \left[\left(a_m^{(m)} - a_{m+1}^{(m)}\right)^2 - 4b_{m+1}^{(m)}\right]^{1/2}\right\}, \tag{8e}$$

and so on.

Corollary 1.2a. *If (5) holds—entailing (6) and (7) with (8)—the polynomial* $p_n^{(n-1)}(x)$ *has the zero* $-a_{n-1}^{(n-1)}$*,*

$$p_n^{(n-1)}\left(-a_{n-1}^{(n-1)}\right) = 0, \tag{9a}$$

and the polynomial $p_n^{(n-2)}(x)$ *has the two zeros* $x_{n-2}^{(\pm)}$*, see (8e),*

$$p_n^{(n-2)}\left(x_{n-2}^{(\pm)}\right) = 0. \tag{9b}$$

The first of these results is a trivial consequence of (6); the second is evident from (7) and (8d). Note moreover that from the factorization formula (7) one can likewise find explicitly three zeros of $p_n^{(n-3)}(x)$ *and four zeros of* $p_n^{(n-4)}(x)$ *by evaluating from (8)* $\tilde{p}_3^{(-m)}(x)$ *and* $\tilde{p}_4^{(-m)}(x)$ *and by taking advantage of the explicit solvability of algebraic equations of degree 3 and 4.*

Corollary 1.2b. *If (5) holds—entailing (6) and (7) with (8)—and moreover the quantities* $a_n^{(m)}$ *and* $b_n^{(m)}$ *satisfy the symmetry properties*

$$a_{n-m}^{(-m)} = a_n^{(m)}, \qquad b_{n-m}^{(-m)} = b_n^{(m)}, \tag{10}$$

then

$$\tilde{p}_n^{(m)}(x) = p_n^{(m)}(x), \tag{11}$$

entailing that factorization (7) takes the neat form

$$p_n^{(m)}(x) = p_{n-m}^{(-m)}(x)\,p_m^{(m)}(x), \qquad m = 0, 1, \ldots, n. \tag{12}$$

The following remark is relevant when both propositions 1.1 and 1.2 hold.

Remark 1.2c. As implied by (3*b*), condition (5) can be enforced via the assignment

$$\alpha^{(\nu)} = A_{\nu-1}^{(\nu-1)} - A_\nu^{(\nu)}, \tag{13}$$

entailing that the nonlinear recursion relation (3*a*) reads

$$\left[A_{n-1}^{(\nu)} - A_{n-1}^{(\nu-1)}\right]\left[A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + A_{\nu-1}^{(\nu-1)} - A_\nu^{(\nu)}\right] = \left[A_{n-1}^{(\nu-1)} - A_{n-1}^{(\nu-2)}\right]\left[A_{n-1}^{(\nu-1)} - A_{n-2}^{(\nu-2)} + A_{\nu-2}^{(\nu-2)} - A_{\nu-1}^{(\nu-1)}\right]. \tag{14}$$

Proposition 1.3. *If the (monic, orthogonal) polynomials* $p_n^{(\nu)}(x)$ *are defined by the three-term recursion relations (1) with coefficients* $a_n^{(\nu)}$ *and* $b_n^{(\nu)}$ *satisfying the requirements sufficient for* *the validity of both propositions 1.1 and 1.2 (namely, (3) with (2) and (5), or just with (14)), then*

$$p_n^{(n)}(x) = \prod_{m=1}^{n} (x - x_m), \tag{15a}$$

with the following (n-independent!) expression of the n zeros x_m:

$$x_m = A_{m-1}^{(m-1)} - A_m^{(m)} \tag{15b}$$

or equivalently (see (3a) and (4b))

$$x_m = -\left(a_{m-1}^{(m-1)} + g_m^{(m)}\right). \tag{15c}$$

The following results are immediate consequences of this proposition 1.3 and of corollary 1.2a.

Corollary 1.3a. *If proposition 1.3 holds, then also the polynomials $p_n^{(n-1)}(x)$ and $p_n^{(n-2)}(x)$ (in addition to $p_n^{(n)}(x)$, see (15)) can be written explicitly:*

$$p_n^{(n-1)}(x) = \left(x + a_{n-1}^{(n-1)}\right) \prod_{m=1}^{n-1} (x - x_m), \tag{16a}$$

$$p_n^{(n-2)}(x) = \left[\left(x + a_{n-1}^{(n-2)}\right)\left(x + a_{n-2}^{(n-2)}\right) + b_{n-1}^{(n-2)}\right] \cdot \prod_{m=1}^{n-2} (x - x_m). \tag{16b}$$

2. Results

Our first task is to show how the machinery of [4] is applicable to the three specific classes of polynomials $p_n^{(\nu)}(x)$, $\tilde{p}_n^{(\nu)}(x)$ and $s_n^{(\nu)}(x)$ introduced in [3], yielding again the properties of these polynomials reported there and reviewed below.

The first class of these (monic, orthogonal) polynomials is defined by the recursion relation [3, equation (8)],

$$p_{n+1}^{(\nu)}(x) = [x + 2n(n-\nu) + 2n + 1]p_n^{(\nu)}(x) - n^2(n-\nu)^2 p_{n-1}^{(\nu)}(x), \tag{17a}$$

$$p_{-1}^{(\nu)}(x) = 0, \qquad p_0^{(\nu)}(x) = 1, \tag{17b}$$

and we now prove again, for these orthogonal polynomials, the factorizations

$$p_n^{(n)}(x) = \prod_{m=1}^{n} [x - m(m-1)], \tag{18a}$$

$$p_n^{(n-1)}(x) = (x+n)p_{n-1}^{(n-1)}(x) = (x+n)_{m=1}^{n-1}[x - m(m-1)], \tag{18b}$$

$$p_n^{(n-2)}(x) = [x^2 + 2(2n-1)x + 2n(n-1)]p \prod_{n-2}^{(n-2)} (x)$$

$$= \left[x - x_n^{(+)}\right]\left[x - x_n^{(-)}\right] \prod_{m=1}^{n-2} [x - m(m-1)], \tag{18c}$$

$$x_n^{(\pm)} = -2n + 1 \pm [n^2 + (n-1)^2]^{1/2} \tag{18d}$$

([3, equations (7) and (14)–(15)]), as well as the formula

$$p_n^{(j)}[k(k-1)] = 0 \qquad \text{if} \quad k = 1, 2, \ldots, j \qquad j = 1, 2, \ldots, n \tag{19}$$

([3, equation (13)]).

Likewise the second of these three classes of orthogonal polynomials is defined by the recursion relation [3, equation (24)],

$$\tilde{p}_{n+1}^{(\nu)}(x) = [x + 2(n+1)(n-\nu)]\tilde{p}_n^{(\nu)}(x) - n(n+1)(n-\nu)(n-\nu-1)\tilde{p}_{n-1}^{(\nu)}(x), \tag{20a}$$

$$\tilde{p}_{-1}^{(\nu)}(x) = 0, \qquad \tilde{p}_0^{(\nu)}(x) = 1, \tag{20b}$$

and we now prove again, for these orthogonal polynomials, the factorizations

$$\tilde{p}_n^{(n)}(x) = \prod_{m=1}^{n}[x - m(m+1)], \tag{21a}$$

$$\tilde{p}_n^{(n-1)}(x) = x\tilde{p}_{n-1}^{(n-1)}(x) = x\prod_{m=1}^{n-1}[x - m(m+1)], \tag{21b}$$

$$\tilde{p}_n^{(n-2)}(x) = x(x+2n)\tilde{p}\prod_{n-2}^{(n-2)}(x) = x(x+2n)_{m=1}^{n-2}[x - m(m+1)] \tag{21c}$$

([3, equations (23) and (30)]), as well as the formulas

$$\tilde{p}_n^{(j)}[k(k+1)] = 0 \qquad \text{if} \quad k = 1, 2, \ldots, j, \qquad j = 1, 2, \ldots, n, \tag{22a}$$

$$\tilde{p}_n^{(j)}(0) = 0 \qquad \text{for} \quad j = 0, 1, \ldots, n-1 \tag{22b}$$

([3, equations (29)]).

And for the third of these three classes of orthogonal polynomials, defined by the recursion relation

$$s_{n+1}^{(\nu)}(x) = xs_n^{(\nu)}(x) + n(n-\nu)s_{n-1}^{(\nu)}(x), \qquad s_0^{(\nu)}(x) = 1 \tag{23}$$

[3, equation (44)], we now prove again the *Diophantine* factorization

$$s_n^{(n)}(x) = \prod_{m=1}^{n}[x - (2m - n - 1)] \tag{24}$$

[3, equation (43)]).

It is indeed plain that the polynomials $p_n^{(\nu)}(x)$ defined by the recursion relation (17) coincide with the polynomials defined by the recursion relations (1) provided

$$a_n^{(\nu)} = 2n(n-\nu) + 2n + 1, \qquad b_n^{(\nu)} = -n^2(n-\nu)^2. \tag{25}$$

Hence they are included in the class of polynomials denoted by $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$ in [4, section 3.1] and characterized by the following assignments sufficient [4] to guarantee the validity of the propositions 1.1, 1.2 and 1.3:

$$a_n^{(\nu)} = k_1 + k_2 + k_3 + \left(-\tfrac{3}{2}k_3 + k_4\right)\nu + [2k_2 + 3k_3(1-\nu)]n + 3k_3n^2, \tag{26a}$$

$$b_n^{(\nu)} = -\tfrac{1}{4}n(3k_3n - 2k_4)[2k_5 + 2(2k_2 + k_4)(n-\nu) + 3k_3(n-\nu)^2], \tag{26b}$$

and entailing via (4*b*)

$$g_n^{(\nu)} = -\tfrac{1}{2}n(3k_3 n - 2k_4). \tag{26c}$$

This inclusion is clearly achieved via the following identifications:

$$k_1 = \tfrac{1}{3}, \qquad k_2 = 0, \qquad k_3 = \tfrac{2}{3}, \qquad k_4 = 0, \qquad k_5 = 0. \tag{27}$$

Hence these polynomials $p_n^{(\nu)}(x)$ satisfy the three propositions 1.1, 1.2 and 1.3, with

$$g_n^{(\nu)} = -n^2. \tag{28}$$

It is likewise plain that the polynomials $\tilde{p}_n^{(\nu)}(x)$ defined by the recursion relation (20) coincide with the polynomials defined by the recursion relation (1) provided

$$a_n^{(\nu)} = 2(n+1)(n-\nu), \qquad b_n^{(\nu)} = -n(n+1)(n-\nu)(n-1-\nu). \tag{29}$$

And they are also included in class (26), with the following identifications:

$$k_1 = -\tfrac{2}{3}, \qquad k_2 = 0, \qquad k_3 = \tfrac{2}{3}, \qquad k_4 = -1, \qquad k_5 = 0. \tag{30}$$

Hence they satisfy as well the three propositions 1.1, 1.2 and 1.3, with

$$g_n^{(\nu)} = -n(n+1). \tag{31}$$

The fact that these two classes of polynomials, $p_n^{(\nu)}(x)$ respectively $\tilde{p}_n^{(\nu)}(x)$ (see (17) respectively (20)), satisfy the three propositions 1.1, 1.2 and 1.3 provides an (additional) proof of the results of [3] relative to these polynomials. In particular: (15) with (25) and (28) respectively with (29) and (31) imply (18*a*) respectively (21*a*); (7) with (18*a*) respectively with (21*a*) imply (19) respectively (22); (16*a*) with (25) respectively with (29) imply (18*b*) respectively (21*b*); (16*b*) with (25) respectively with (29) imply (18*c*) respectively (21*c*).

As for the polynomials $s_n^{(\nu)}(x)$ of [3], see (23), they clearly coincide with the polynomials defined by the recursion relation (1) provided

$$a_n^{(\nu)} = 0, \qquad b_n^{(\nu)} = n(n-\nu). \tag{32}$$

It is therefore easily seen that they are *not* included in class (26): this was expected, since the polynomials of class (26) yield 'Diophantinely factorized' polynomials $p_n^{(\nu)}(x)$ whose zeros are *independent* of the order of the polynomial, see (15), while the polynomials $s_n^{(\nu)}(x)$ yield 'Diophantinely factorized' polynomials $s_n^{(n)}(x)$ whose n zeros $2m-n-1$, see (24), clearly do depend on the order n of the polynomial. To include them within the treatment of [4] one needs an additional twist, that we now describe.

We introduce the following more general family of polynomials $\sigma_n^{(\nu,k)}(x)$ defined (for arbitrary values of the two parameters ν and k, and for all *positive* integer values of n) by the recursion relation

$$\sigma_{n+1}^{(\nu,k)}(x) = (x+k-\nu)\sigma_n^{(\nu,k)}(x) + n(n-\nu)\sigma_{n-1}^{(\nu,k)}(x), \qquad \sigma_0^{(\nu,k)}(x) = 1. \tag{33a}$$

A comparison of this recursion relation to (23) clearly entails that

$$\sigma_n^{(\nu,\nu)}(x) = s_n^{(\nu)}(x). \tag{33b}$$

On the other hand, it is clear that the polynomials defined by the recursion relation (33*a*) coincide with the polynomials defined by the recursion relation (1) provided

$$a_n^{(\nu)} = k-\nu, \qquad b_n^{(\nu)} = n(n-\nu). \tag{34}$$

And it is now plain that the polynomials $\sigma_n^{(\nu,k)}(x)$ (with k considered as a fixed parameter) are now included in class (26), with the following identification:

$$k_1 = k, \qquad k_2 = 0, \qquad k_3 = 0, \qquad k_4 = -1, \qquad k_5 = 0. \tag{35}$$

Hence these polynomials $\sigma_n^{(\nu,k)}(x)$ (with k considered as a fixed parameter) satisfy the three propositions 1.1, 1.2 and 1.3 with

$$g_n^{(\nu)} = -n \tag{36}$$

(see (26c)), and this entails for $\nu = n$, via (15), the *Diophantine* factorization

$$\sigma_n^{(n,k)}(x) = \prod_{m=1}^{n} [x - (2m - k - 1)]. \tag{37}$$

For $k = n$ this yields, via (33b), the Diophantine factorization (24).

This completes our first task.

Our second task is to use the machinery of [4] to provide a proof of the findings due to Christophe Smet, see below (51), (52) and (54), corresponding to the *additional remark* at the end of section 1 of [3]—as well as *new* analogous *Diophantine* results. To this end the following two *propositions* play a crucial role—besides constituting findings of interest in their own right.

Proposition 2.1. *Assume that the class of (monic, orthogonal) polynomials* $p_n^{(\nu)}(x)$ *of [4], defined by recursion relation (1), satisfies proposition 1.1, hence they also obey the (second) recursion relation*

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)} p_{n-1}^{(\nu-1)}(x) \tag{38}$$

(see (4)). Then there also holds the relations

$$p_n^{(\nu)}(x) = \left[x - x_n^{(1,\nu)}\right] p_{n-1}^{(\nu-1)}(x) + b_{n-1}^{(\nu-1)} p_{n-2}^{(\nu-1)}(x), \tag{39a}$$

$$x_n^{(1,\nu)} = -\left[a_{n-1}^{(\nu-1)} + g_n^{(\nu)}\right], \tag{39b}$$

as well as

$$p_n^{(\nu)}(x) = \left[x - x_n^{(2,\nu)}\right] p_{n-1}^{(\nu-2)}(x) + c_n^{(\nu)} p_{n-2}^{(\nu-2)}(x), \tag{40a}$$

$$x_n^{(2,\nu)} = -\left[a_{n-1}^{(\nu-2)} + g_n^{(\nu)} + g_n^{(\nu-1)}\right], \tag{40b}$$

$$c_n^{(\nu)} = b_{n-1}^{(\nu-2)} + g_n^{(\nu)} g_{n-1}^{(\nu-1)}, \tag{40c}$$

as well as

$$p_n^{(\nu)}(x) = \left[x - x_n^{(3,\nu)}\right] p_{n-1}^{(\nu-3)}(x) + d_n^{(\nu)} p_{n-2}^{(\nu-3)}(x) + e_n^{(\nu)} p_{n-3}^{(\nu-3)}(x), \tag{41a}$$

$$x_n^{(3,\nu)} = -\left[a_{n-1}^{(\nu-3)} + g_n^{(\nu)} + g_n^{(\nu-1)} + g_n^{(\nu-2)}\right], \tag{41b}$$

$$d_n^{(\nu)} = b_{n-1}^{(\nu-3)} + g_n^{(\nu)} g_{n-1}^{(\nu-2)} + g_n^{(\nu-1)} g_{n-1}^{(\nu-2)} + g_n^{(\nu)} g_{n-1}^{(\nu-1)}, \tag{41c}$$

$$e_n^{(\nu)} = g_n^{(\nu)} g_{n-1}^{(\nu-1)} g_{n-2}^{(\nu-2)}. \tag{41d}$$

Proof. To obtain (39a), apply the recursion relation (1) to lower the index n of the first polynomial on the right-hand side of (38) and use definition (39b). To obtain (40), apply the recursion relation (38) to the two polynomials on the right-hand side of (38) (thereby decreasing by one unit their upper index, from $\nu - 1$ to $\nu - 2$), then apply the recursion relation (1) to the polynomial $p_n^{(\nu-2)}(x)$ on the right-hand side of the resulting formula, to lower its order n. This yields (40). To obtain (41), apply the recursion (38) to the two polynomials on the right-hand side of (40*a*) (thereby decreasing by one unit their upper index, from $\nu-2$ to $\nu-3$); this yields (41). □

Proposition 2.2. *Assume that for the class of polynomials* $p_n^{(\nu)}(x)$ *there holds the preceding proposition 2.1, and moreover that, for some value of the parameter* μ *(and of course for all nonnegative integer values of* n*),*

$$b_n^{(n+\mu)}=0 \tag{42}$$

(see (1) and (39)), then the polynomials $p_n^{(n+\mu)}(x)$ *factorize as follows:*

$$p_n^{(n+\mu)}(x)=\prod_{m=1}^{n}\left[x-x_m^{(1,m+\mu)}\right] \tag{43a}$$

(using the standard convention according to which a product equals unity when its lower limit exceeds its upper limit), so that

$$p_0^{(\mu)}(x)=1,\qquad p_1^{(1+\mu)}(x)=x-x_1^{(1,1+\mu)},\qquad p_2^{(2+\mu)}(x)=\left[x-x_1^{(1,2+\mu)}\right]\left[x-x_2^{(1,2+\mu)}\right], \tag{43b}$$

and so on. Likewise if, for all non-negative integer values of n*, the coefficients* $c_n^{(2n+\mu)}$ *vanish (see (40)),*

$$c_n^{(2n+\mu)}=b_{n-1}^{(2n+\mu-2)}+g_n^{(2n+\mu)}g_{n-1}^{(2n+\mu-1)}=0, \tag{44}$$

then the polynomials $p_n^{(2n+\mu)}(x)$ *factorize as follows:*

$$p_n^{(2n+\mu)}(x)=\prod_{m=1}^{n}\left[x-x_m^{(2,2m+\mu)}\right], \tag{45a}$$

entailing

$$p_0^{(\mu)}(x)=1,\qquad p_1^{(2+\mu)}(x)=x-x_1^{(2,2+\mu)},\qquad p_2^{(4+\mu)}(x)=\left[x-x_1^{(2,2+\mu)}\right]\left[x-x_2^{(2,4+\mu)}\right], \tag{45b}$$

and so on. And thirdly if, for all non-negative integer values of n*, the following two properties hold (see (41)):*

$$d_n^{(3n+\mu)}=b_{n-1}^{(3n+\mu-3)}+g_n^{(3n+\mu)}g_{n-1}^{(3n+\mu-2)}+g_n^{(3n+\mu-1)}g_{n-1}^{(3n+\mu-2)}+g_n^{(3n+\mu)}g_{n-1}^{(3n+\mu-1)}=0, \tag{46a}$$

$$e_n^{(3n+\mu)}=0 \quad \text{i.e. } g_n^{(3n+\mu)}=0 \quad\text{or}\quad g_{n-1}^{(3n+\mu-1)}=0 \quad\text{or}\quad g_{n-2}^{(3n+\mu-2)}=0, \tag{46b}$$

then the polynomials $p_n^{(3n+\mu)}(x)$ *factorize as follows:*

$$p_n^{(3n+\mu)}(x)=\prod_{m=1}^{n}\left[x-x_m^{(3,3m+\mu)}\right], \tag{47a}$$

entailing

$$p_0^{(\mu)}(x)=1,\qquad p_1^{(3+\mu)}(x)=x-x_1^{(3,3+\mu)},\qquad p_2^{(6+\mu)}(x)=\left[x-x_1^{(3,3+\mu)}\right]\left[x-x_2^{(3,6+\mu)}\right], \tag{47b}$$

and so on.

Here of course the n *(*n*-independent!) zeros* $x_m^{(1,m+\mu)}$, $x_m^{(2,2m+\mu)}$ *respectively* $x_m^{(3,3m+\mu)}$*, are defined by (39b), (40b) respectively (41b)—this often entails the Diophantine character of these findings, in as much as these zeros are given by neat formulas in terms of integers.*

Proof. (39*a*) with (42) entail

$$p_n^{(n+\mu)}(x) = \left[x - x_n^{(1,n+\mu)}\right] p_{n-1}^{(n+\mu-1)}(x), \qquad p_0^{(n+\mu)}(x) = 1, \tag{48}$$

and this two-term recursion relation (in the index n) is clearly solved by formula (43*a*). Likewise (40*a*) with (44) entails

$$p_n^{(2n+\mu)}(x) = \left[x - x_n^{(2n+\mu)}\right] p_{n-1}^{(2(n-1)+\mu)}(x), \tag{49}$$

and this two-term recursion relation in n is clearly solved by formula (45*a*) (note incidentally that (45*b*) with $n = 1$ is consistent with equation (1*c*) of [4] since, via equation (85*d*) of [4], $x_1^{(2+\mu)} = -a_0^{(2+\mu)}$). Finally, (41*a*) with (46*a*) and (46*b*) entails

$$p_n^{(3n+\mu)}(x) = \left[x - x_n^{(3n+\mu)}\right] p_{n-1}^{(3(n-1)+\mu)}(x), \tag{50}$$

and this two-term recursion relation in n is clearly solved by formula (47*a*). □

Remark 2.3. Clearly, the procedure that yielded the *three* factorizations (43), (45) and (47) could be further implemented, obtaining thereby other analogous factorization formulas, for whose validity however additional conditions are required, so that their interest becomes more and more marginal. Indeed, we have not found so far any nontrivial class of orthogonal polynomials satisfying the two conditions (46) yielding the factorization (47).

Remark 2.4. This proposition 2.2 provides a significant extension of the propositions 1.2 and 1.3. And it is plain that, for the polynomials $p_n^{(\nu)}(x)$ respectively $\tilde{p}_n^{(\nu)}(x)$ introduced in [3] (note the validity for $\mu = 1$ of (44) with (25) and (28) respectively with (29) and (31)) this proposition 2.2 demonstrates the validity of the *Diophantine* formulas—whose validity was numerically observed by Christophe Smet [3]—that we set out to prove

$$p_n^{(2n+1)}(x) = \prod_{m=1}^{n} [x - 2m(2m-1)], \tag{51}$$

$$\tilde{p}_n^{(2n+1)}(x) = \prod_{m=1}^{n} [x - 2m(2m+1)], \tag{52}$$

see (45) with (40*b*) and with (25) and (28) respectively with (29) and (31). It is moreover plain (note the validity for $\mu = -1$ of (42) with (29)) that there holds the following *new* Diophantine finding (see (43) with (39*b*), (29) and (31)):

$$\tilde{p}_n^{(n-1)}(x) = \prod_{m=1}^{n} [x - m(m-1)]. \tag{53}$$

Finally, let us provide an analogous *proof*, utilizing the results of [4], of the *Diophantine* factorization formula

$$s_n^{(2n+1)}(x) = \prod_{m=1}^{n} [x - 2(2m-n-1)], \tag{54}$$

whose existence was noticed by Christophe Smet; see the *additional remark* reported at the end of the introductory section of [3]. To this end we use again the polynomials $\sigma_n^{(\nu,k)}(x)$ defined by the recursion relation (33*a*), as well as the fact that these polynomials $\sigma_n^{(\nu,k)}(x)$ (with k a fixed parameter) are included in class (26) with assignment (35), hence satisfy the three propositions 1.1, 1.2 and 1.3. It is then plain that for these polynomials (see (40*c*) with (34) and (36))

$$c_n^{(\nu)} = (1-\mu)(n-1), \tag{55}$$

hence, for $\mu = 1$ (see (44)), they satisfy proposition 2.2 entailing (see (45) with (40*b*), (34), (36) and of course $\mu = 1$)

$$\sigma_n^{(2n+1,k)}(x) = \prod_{m=1}^{n} [x - (4m - k - 1)]. \tag{56}$$

For $k = 2n + 1$ this yields, via (33*b*), the *Diophantine* factorization (54).

3. Outlook

In the next paper [8] of this series the machinery developed in previous papers [3, 4] and in this one is applied to the orthogonal polynomials of the Askey–Wilson scheme (see for instance [9]). And a referee kindly pointed out the likelihood and desirability that our findings be also extended to the q-case, mentioning that some results in this direction are already provided in the classical paper by Askey and Wilson [10].

Appendix A

In this appendix, we prove again certain results concerning the polynomials $p_n^{(\nu)}(x)$, $\tilde{p}_n^{(\nu)}(x)$ and $s_n^{(\nu)}(x)$ introduced in [3], including in particular the findings due to Christophe Smet (see the *additional remark* at the end of section 1 of that paper, and section 2 above). These proofs are more ad hoc and more cumbersome than those presented above; we nevertheless report them here because it is of some interest to also present proofs performed by a quite different route, which moreover includes the derivation of two identities involving binomial coefficients that we did not find in the standard literature [5–7] hence might be of some interest in their own right.

Firstly let us observe that, for $\nu = n$, the hypergeometric function appearing on the right-hand side of equation (31*a*) of [3], reading

$$p_n^{(\nu)}(x) = n!(1-\nu)_n \, {}_3F_2(-n, m_+(x), m_-(x); 1, 1-\nu; 1), \tag{A.1}$$

can be summed via the Saalschütz formula, see [5, equation 4.4(3)] with $a = m_+(x)$, $b = m_-(x) = 1 - a$, $c = 1$, yielding

$$\begin{aligned} p_n^{(n)}(x) &= n!(1-n)_n \frac{(1-m_+)_n(1-m_-)_n}{n!(-1)_n} \\ &= (-1)^n \prod_{m=1}^{n} \{[m - m_+(x)][m - m_-(x)]\} \\ &= (-1)^n \prod_{\ell=1}^{n} \left\{ \left(m - \frac{1}{2}\right)^2 - \frac{1+4x}{4} \right\} \\ &= \prod_{\ell=1}^{n} [x - m(m-1)], \end{aligned} \tag{A.2}$$

which of course reproduces [3, equation (7a)]. Here and below $m_\pm \equiv m_\pm(x) = [1 \pm (1+4x)^{1/2}]/2$ [3, equation (31b)]), and the only slightly nontrivial formula we used above is the identity $(1-n)_n/(-1)_n = (-1)^n$.

The second observation is that, for ν *integer* and larger than n, say $\nu = N > n$, formula (A.1) ceases to hold. Indeed, in such a case the (standard representation of the) hypergeometric function on the right-hand side of this formula consists of a finite sum from 0 to n (which indeed corresponds to the left-hand side), and of a second sum from N to ∞, which generally diverges. This can be explicitly seen for $\nu = 2n + 1$, when the hypergeometric sum on the right-hand side of this formula can be exactly summed using *both* Watson's and Whipple's formulas, see [5, equations 4.4(6) and 4.4(7)]. The simultaneous applicability in this case of both these formulas suggests that this value of ν is special, as already noted by Christophe Smet who—on the basis of numerical evidences for small values of n—conjectured the validity for all (*non-negative integer*) values of n of (51) [3, formula (A.3)]). Let us now prove (again!: see above) this formula, or rather let us prove that

$$p_n^{(2n+1)}[2k(2k-1)] = 0, \qquad k = 1, \ldots, n, \tag{A.3}$$

since this formula, together with the fact that the polynomial $p_n^{(2n+1)}(x)$ is *monic*, entails (51).

Inserting $2k(2k-1)$ in place of x, and $2n+1$ in place of ν, on the right-hand side of the formula

$$p_n^{(\nu)}(x) = p_n^{(n)}(x) + \sum_{m=0}^{n-1}\left\{\left(\frac{n!}{m!}\right)^2 \frac{p_m^{(m)}(x)}{(n-m)!}\prod_{\ell=m+1}^{n}(\ell-\nu)\right\}, \tag{A.4a}$$

with

$$p_n^{(n)}(x) = \prod_{m=1}^{n}[x - m(m-1)] \tag{A.4b}$$

(equation (9) with (7a) of [3]), and noting that the second of these formulae entails

$$p_m^{(m)}[2k(2k-1)] = \frac{(2k-1+m)!}{(2k-1-m)!}, \tag{A.5}$$

one easily gets, after some standard developments,

$$p_n^{(2n+1)}[2k(2k-1)] = \sum_{m=0}^{\min(n,2k-1)}\left[(-1)^{n-m}\left(\frac{n!}{m!}\right)\frac{(2n-m)!(2k+m-1)!}{m!(n-m)!(2k-m-1)!}\right]. \tag{A.6}$$

Here and hereafter k (as well of course as n) is a *non-negative integer*. Via the identity

$$\frac{q!}{(q-x)!} = \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^x z^q|_{z=1}, \qquad x = 0, 1, \ldots, q, \qquad q = 0, 1, 2, \ldots, \tag{A.7}$$

this formula can be rewritten in the following two ways, provided

$$k \leqslant n \tag{A.8}$$

as we hereafter assume

$$p_n^{(2n+1)}[2k(2k-1)] = \left(\frac{\partial}{\partial x}\right)^{2k-1}\left(\frac{\partial}{\partial y}\right)^{2n-2k+1}\sum_{m=0}^{\min(n,2k-1)} \times\left[(-1)^{n-m}\binom{n}{m}x^{2k-1+m}y^{2n-m}\right]\Bigg|_{x=y=1}, \tag{A.9a}$$

$$p_n^{(2n+1)}[2k(2k-1)] = \frac{n!}{(2k-1)!}\left(\frac{\partial}{\partial x}\right)^{2k-1}\left(\frac{\partial}{\partial y}\right)^{n}\sum_{m=0}^{\min(n,2k-1)} \times\left[(-1)^{n-m}\binom{2k-1}{m}x^{2k-1+m}y^{2n-m}\right]\Bigg|_{x=y=1}. \tag{A.9b}$$

Here and hereafter the symbol $\binom{n}{m}$ denotes the binomial coefficient

$$\binom{n}{m} \equiv \frac{n!}{m!(n-m)!}, \qquad m = 0, 1, \ldots, n, \qquad n = 0, 1, 2 \ldots. \tag{A.10}$$

Let us now assume firstly that

$$2k - 1 \geqslant n. \tag{A.11}$$

Then clearly (A.9*a*) reads

$$p_n^{(2n+1)}[2k(2k-1)] = \left(\frac{\partial}{\partial x}\right)^{2k-1} \left(\frac{\partial}{\partial y}\right)^{2n-2k+1} x^{2k-1} y^n (x-y)^n \bigg|_{x=y=1}. \tag{A.12}$$

Next, we use the identity

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^p f(z)g(z) = \sum_{\ell=0}^{p} \binom{p}{\ell} \left[\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{p-\ell} f(z)\right]\left[\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{\ell} g(z)\right], \qquad x = 0, 1, 2, \ldots \tag{A.13}$$

to rewrite this formula as follows:

$$p_n^{(2n+1)}[2k(2k-1)] = \sum_{\ell=0}^{2k-1} \left[(-1)^\ell \binom{2k-1}{\ell} \binom{2n-(2k-1)}{n-\ell}\right]. \tag{A.14}$$

Note that to reduce the double sum to a single sum we took advantage of the fact that $(x-y)^p$ vanishes for $x = y$ unless $p = 0$. It is then plain that the right-hand side of this formula vanishes, since by replacing in the sum the dummy summation integer ℓ with $2k-1-\ell$, and using the standard identities

$$\binom{2k-1}{2k-1-\ell} = \binom{2k-1}{\ell}, \qquad \binom{2n-(2k-1)}{n-(2k-1)+\ell} = \binom{2n-(2k-1)}{n-\ell} \tag{A.15}$$

one easily sees that the same sum gets reproduced up to a factor $(-1)^{2k-1} = -1$. Hence we conclude that $p_n^{(2n+1)}[2k(2k-1)]$ vanishes provided the two inequalities (A.8) and (A.11) hold.

Let us now assume that the second of these two inequalities does *not* hold, namely that

$$2k - 1 < n. \tag{A.16}$$

Then we start from (A.9*b*) and proceeding as above we arrive now at the formula

$$p_n^{(2n+1)}[2k(2k-1)] = \sum_{\ell=0}^{2k-1} \left[(-1)^{n-\ell} \binom{2k-1}{\ell} \binom{n}{2k-1-\ell} \binom{n}{\ell}\right]. \tag{A.17}$$

And it is again plain that the right-hand side of this formula vanishes, since it changes sign under the replacement of the dummy index ℓ with $2k-1-\ell$.

Formula (51) is thereby proven.

Remark A1. It is easily seen (by replacing in the right-hand side of (A.6) $2k-1$ with m, and the dummy summation variable m with ℓ) that the result we just proved entails the identity

$$\sum_{\ell=0}^{\min(n,m)} \left[(-1)^\ell \binom{2\ell}{\ell} \binom{m+\ell}{2\ell} \binom{2n-\ell}{\ell}\right] = 0, \qquad n = 0, 1, 2, \ldots, \qquad m = 1, 3, 5, \ldots. \tag{A.18}$$

We were unable to find this remarkable formula in the standard literature [5–7].

The analog of formula (51) for the tilded polynomials $\tilde{p}_n^{(\nu)}(x)$ of [3] is (52), and its proof via this route is sufficiently analogous to that given above for (51) that it can be left to the diligent reader.

Finally, let us prove (again!: see above) formula (54), namely

$$s_n^{(2n+1)}[2(2k-n-1)] = 0, \qquad k = 1, 2, \ldots, n. \tag{A.19}$$

Now our starting point is formula [3, equation (46)],

$$s_n^{(\nu)}(x) = \hat{s}_n^{(\nu)}(x) + \sum_{m=0}^{n-1} \left\{ \frac{n!}{m!} \frac{\hat{s}_m^{(\nu)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\} \tag{A.20a}$$

with

$$\hat{s}_0^{(\nu)}(x) = 1, \qquad \hat{s}_n^{(\nu)}(x) = \prod_{m=1}^{n} [x - (2m - \nu - 1)], \qquad n = 1, 2, \ldots. \tag{A.20b}$$

This entails (after some standard developments, analogous to those performed above)

$$\begin{aligned} s_n^{(2n+1)}[2(2k-n-1)] &= (-1)^n \sum_{m=0}^{\min(n,2k-1)} \left[(-2)^m \frac{(2k-1)!(2n-m)!}{m!(n-m)!(2k-1-m)!} \right] \\ &= (-1)^n n! \sum_{m=0}^{\min(n,2k-1)} \left[(-2)^m \binom{2k-1}{m} \binom{2n-m}{n-m} \right]. \end{aligned} \tag{A.21}$$

Let us firstly assume that

$$n \leqslant 2k - 1 \leqslant 2n. \tag{A.22}$$

Then via (A.7) and (A.13) this formula, (A.21), yields

$$\begin{aligned} s_n^{(2n+1)}[2(2k-n-1)] &= (-1)^n \frac{(2k-1)!}{n!} \left(\frac{d}{dz} \right)^{2n-2k+1} \left\{ z^{2n} \sum_{m=0}^{n} \left[\binom{n}{m} \left(-\frac{2}{z} \right)^m \right] \right\} \Bigg|_{z=1} \\ &= \frac{(2k-1)!}{n!} \left(\frac{d}{dz} \right)^{2n-2k+1} \{ [1 - (z-1)^2]^n \} \Big|_{z=1} \\ &= \frac{(2k-1)!}{n!} \left(\frac{d}{dz} \right)^{2n-2k+1} \left\{ \sum_{\ell=0}^{n} \left[(-1)^\ell \binom{n}{\ell} (z-1)^{2\ell} \right] \right\} \Bigg|_{z=1}, \end{aligned} \tag{A.23}$$

and this clearly vanishes because $2n - 2k + 1$ is odd while 2ℓ is even hence they never coincide, while of course for $z = 1$ the power $(z-1)^p$ vanishes unless $p = 0$.

Next let us assume that $2k - 1 \leqslant n$. Then via (A.7) formula (A.21) yields

$$\begin{aligned} s_n^{(2n+1)}[2(2k-n-1)] &= (-1)^n \left(\frac{d}{dz} \right)^n \left\{ z^{2n} \sum_{m=0}^{2k-1} \left[\binom{2k-1}{m} \left(-\frac{2}{z} \right)^m \right] \right\} \Bigg|_{z=1} \\ &= (-1)^n \left(\frac{d}{dz} \right)^n \{ z^{2n-2k+1} (z-2)^{2k-1} \} \Big|_{z=1} \\ &= (-1)^{n-2k+1} \left(\frac{d}{dz} \right)^n \{ [1 + (z-1)]^{2n-2k+1} [1 - (z-1)]^{2k-1} \} \Big|_{z=1} \\ &= (-1)^{n-2k+1} n! \sum_{\ell=0}^{2k-1} \left[(-1)^\ell \binom{2n-(2k-1)}{n-\ell} \binom{2k-1}{\ell} \right]. \end{aligned} \tag{A.24}$$

The last step has been performed by expanding the two expressions $[1 + (z - 1)]^{2n-2k+1}$ and $[1 - (z - 1)]^{2k-1}$ in powers of $(1 - z)$, then applying the differential operator and using the property that for $z = 1$ the power $(1 - z)^p$ vanishes unless $p = 0$. And clearly the sum on the right-hand side of the last formula vanishes, for the same reason that entailed the vanishing of the right-hand side of (A.14).

Formula (54) is thereby proven.

Remark A2. It is easily seen (by replacing on the right-hand side of (A.21) $2k - 1$ with m, and the dummy summation variable m with ℓ) that the result we just proved entails the identity

$$\sum_{\ell=0}^{\min(n,m)} \left[(-2)^{\ell} \binom{m}{\ell} \binom{2n-\ell}{n-\ell} \right] = 0, \qquad n = 0, 1, 2 \ldots, \qquad m = 1, 3, 5, \ldots. \quad \text{(A.25)}$$

We were unable to find this remarkable formula in the standard literature [5–7].

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